Transport Capacity of Broadcast Ad-Hoc Wireless Networks

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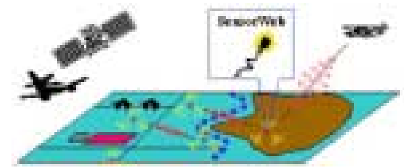
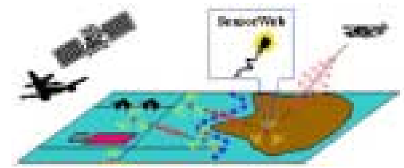


Outline

- Introduction
- “Optimal” Communication Scheme
  - Main result: TC maximizing power allocation
  - Transport Capacity of large networks
- TDMA
  - Compare with optimal Communication Scheme
- Summary/conclusions and directions for future work

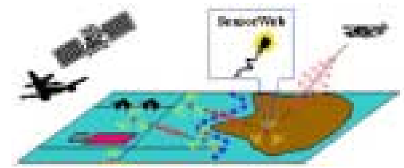


Why Transport Capacity?

- Wireless ad-hoc networks are inherently thought of as located in some region of space

- Thus, instead of just asking “what is the best set of rates that can be delivered?” (Shannon capacity region) one may ask
  - How much data can be delivered per unit distance?
  - What is the most efficient way to cover an area of space or a volume of space?

Gupta and Kumar (2000); Gupta (2000)



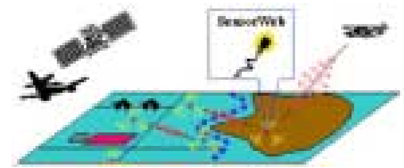
Why Broadcast?

- Simplify the problem significantly: consider a "network" where only one node is transmitting and the rest are receiving - a broadcast scenario

- Advantages
  - Shannon capacity of broadcast channels is known for "degraded broadcast channels"
  - Gaussian broadcast channel is degraded and the general expression simplifies to a nice formula

Cover ('72); Bergmans ('73); Gallager ('74)

Capacity of Gaussian Broadcast Channel

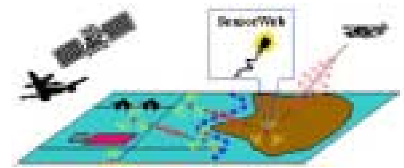


$$R_1 < C\left(\frac{\alpha_1 P}{N_1}\right)$$

. . .

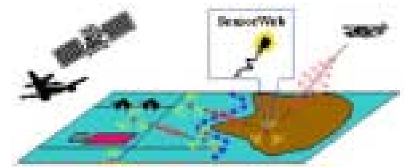
$$R_k < C\left(\frac{\alpha_k P}{N_k + \sum_{i=1}^{k-1} \alpha_i P}\right)$$

- C(·) - single-user Gaussian capacity function
- a_k - proportion of total power allocated to user k
- N_k - noise power at receiver k (require $N_k > N_{k-1}$)



What is Our Problem?

- Based on the distance to the receiver, assign a reward for transmitting to that receiver
- Optimize the total reward over the set of achievable rates
  - for reward functions that are linear functions of rates work on the boundary of Shannon capacity regions
- To solve the optimization problem need to know the Shannon capacity region
- The receiver configuration is fixed



Our Problem

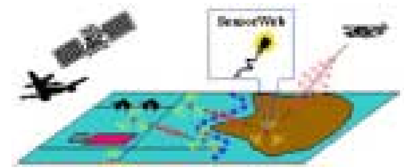
- For the Gaussian Broadcast Channel, maximize

$$TC = \sum_{k=1}^{K} d_k^{\rho} R_k$$

- where

$$0 < d_1 < \ldots < d_k < \ldots < d_K$$

  - ***note:*** no generality is lost in requiring strict inequality since for our purposes 2 equidistant receivers are equivalent to a single receiver to which we allocated the sum of the two rates



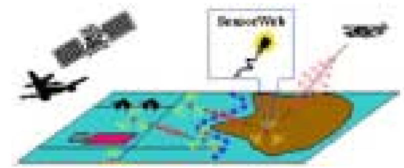
Our Problem

- Optimization variable: tradeoff vector **a**

$$\mathbf{a} = [\alpha_1, \ldots, \alpha_k, \ldots, \alpha_K]$$

  - subject to

$$\alpha_k \geq 0 \ \forall k \ \text{ and } \ \sum_{k=1}^{K} \alpha_k = 1$$



Our Problem

- Also use: cumulative tradeoff vector **b**

$$\mathbf{b} = [\beta_1, \ldots, \beta_k, \ldots, \beta_{K-1}] \quad \text{where} \quad \beta_k = \sum_{i=1}^{k} \alpha_i$$

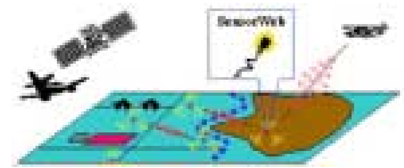
  - subject to

$$\beta_k \geq 0 \quad \forall k$$

$$\beta_k \leq 1 \quad \forall k$$

$$\beta_k \leq \beta_{k+1} \quad k = 1, \ldots, K-1$$

  - and $\beta_K \equiv 1$ (note: β_K is not a variable)



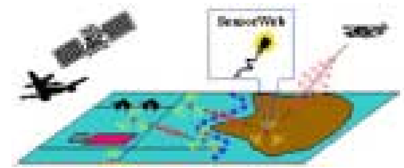
Our Problem

- Subject to
  - "optimal" communication scheme: operate at the boundary of the capacity region

$$R_k = C\left(\frac{\alpha_k P}{N_k + \sum_{i=1}^{k-1} \alpha_i P}\right)$$

  - TDMA: time sharing

$$R_k = \alpha_k C\left(\frac{P}{N_k}\right)$$

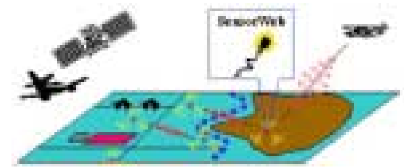


Power Law Channel

- Actual channel
  - Signal power depends only on the distance to the transmitter
  - Signal power decays as γ power of the distance; noise power constant
- Channel model
  - Signal power remains constant (*P*); noise power increases as γ power of the distance

$$N_k = N d_k^{\gamma}$$

  - This makes the channel a degraded broadcast channel

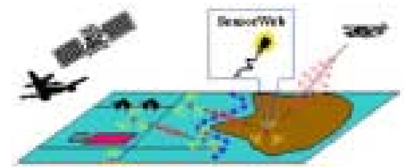


Optimal Communication Scheme

- Note that the rate constraint can be re-written as

$$R_k = \frac{1}{2}\log_2\left(\frac{\frac{N}{P}d_k^{\gamma} + \beta_k}{\frac{N}{P}d_k^{\gamma} + \beta_{k-1}}\right)$$

- Approach: work with this form of the constraint and solve a much more general problem



Optimal Communication Scheme

- Maximize

$$TC = \sum_{k=1}^{K} r(d_k) R_k$$

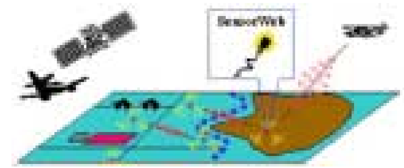
- where

$$R_k = \frac{1}{2}\log_2\left(\frac{s(d_k)+\beta_k}{s(d_k)+\beta_{k-1}}\right)$$

- variable of optimization:

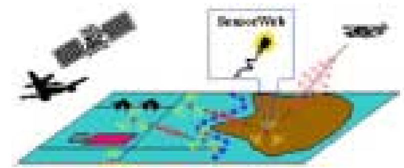
$$\mathbf{b} = \left[\beta_1, \ldots, \beta_k, \ldots, \beta_{K-1}\right]$$

as defined before



Optimal Communication Scheme

- Assumptions:
  - Reward function (*r*)
    - notation: $r_k = r(d_k)$
    - strictly increasing: $r_k > r_{k-1}$
  - Noise-to-signal ratio (NSR) function (*s*)
    - also called the “penalty” function
    - notation: $s_k = s(d_k)$
    - strictly increasing: $s_k > s_{k-1}$
  - $$\frac{s_{k+1} - s_k}{s_k - s_{k-1}} > \frac{r_{k+1} - r_k}{r_k - r_{k-1}} \qquad (*)$$



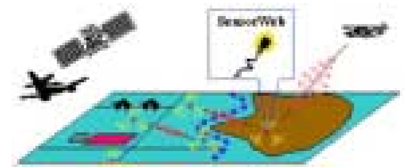
Optimal Communication Scheme

- Sufficient condition for (*) to hold for all choices of distances. If *s*() and *r*() are differentiable everywhere on (0,∞), then (*) holds for all choices of distances if and only

$$\frac{r'(x)}{s'(x)}$$

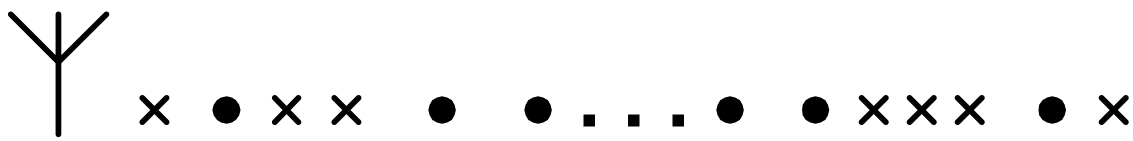
is strictly monotonically decreasing

- Intuition: channel penalty has to grow at a faster rate then the reward given for transmission to a certain distance



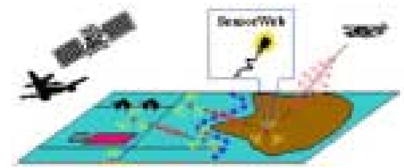
Optimal Communication Scheme:Main Result

Transmit to a “block” of receiver, plus at most one more receiver in front of the block and at most one more receiver behind the block



Proof Strategy:

- Guess the solution
- Show that any other configuration can be improved on



Optimal Communication Scheme

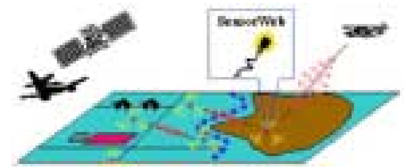
- To arrive at the guess: solve the unconstrained problem
- Solution:

$$\beta_k^* = \frac{s_{k+1} r_k - s_k r_{k+1}}{r_{k+1} - r_k}$$

- “Optimal unconstrained rates” are

$$R_k^* = \frac{1}{2}\log_2\left(\frac{r_k - r_{k-1}}{r_{k+1} - r_k}\frac{s_{k+1} - s_k}{s_k - s_{k-1}}\right)$$

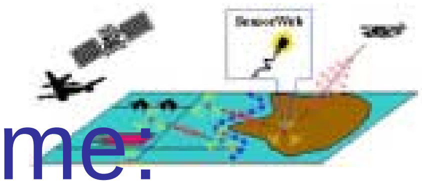
- NOTE: $R^*_k > 0 \Leftrightarrow \beta^*_{k+1} > \beta^*_k \Leftrightarrow$ (*)



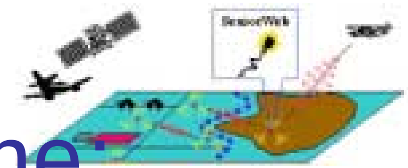
Optimal Communication Scheme

- Unconstrained solution does not satisfy the constraint $\beta_k \geq 0$ or $\beta_k \leq 1$.
- Ad-hoc strategy: transmit only to receivers whose unconstrained optimal β is in [0,1] if there is some leftover power, find something good to do with it
- The actual optimal strategy (our guess) is almost this ad-hoc strategy

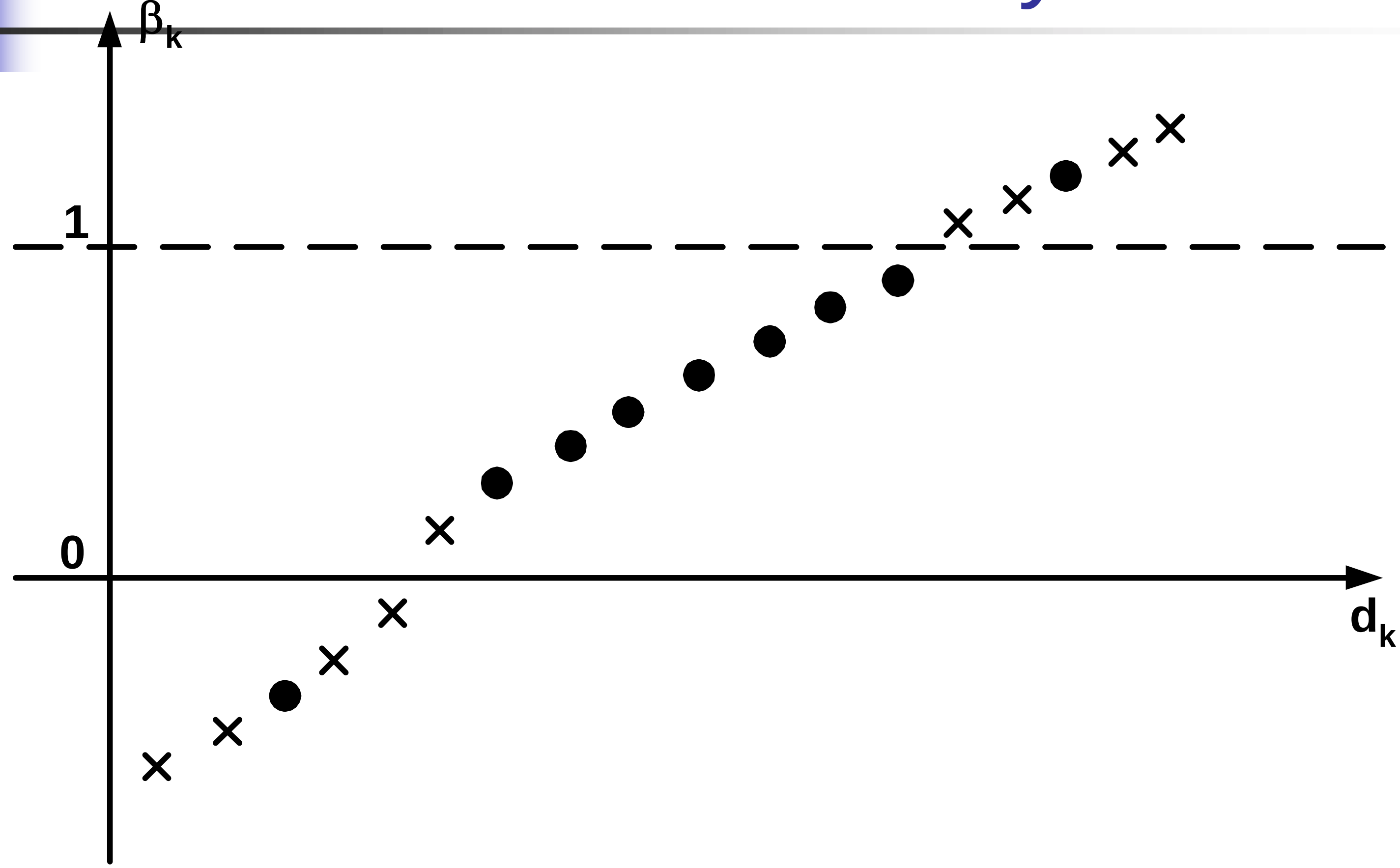
Optimal Communication Scheme: Main Result More Precisely



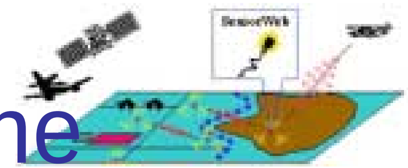
- Transport Capacity optimizing communication strategy
  - transmit to all receivers whose optimal unconstrained β is in [0,1] - except the very first one
  - transmit to at most one additional receiver in front of the “main block”
  - transmit to at most one additional receiver behind the “main block”



Optimal Communication Scheme: Main Result More Precisely



Optimal Communication Scheme Power Law Channel



- (*) if and only if $\gamma>\rho$
- Optimal unconstrained β's are given by

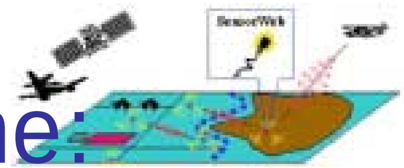
$$\beta_k^* = \frac{N}{P}\frac{d_{k+1}^{\rho} d_k^{\rho}\left(d_{k+1}^{\gamma-\rho} - d_k^{\gamma-\rho}\right)}{d_{k+1}^{\rho} - d_k^{\rho}}$$

- The corresponding rates are given by

$$R_k = \frac{1}{2}\log_2\left(\frac{d_k^{\rho} - d_{k-1}^{\rho}}{d_{k+1}^{\rho} - d_k^{\rho}}\frac{d_{k+1}^{\gamma} - d_k^{\gamma}}{d_k^{\gamma} - d_{k-1}^{\gamma}}\right)$$

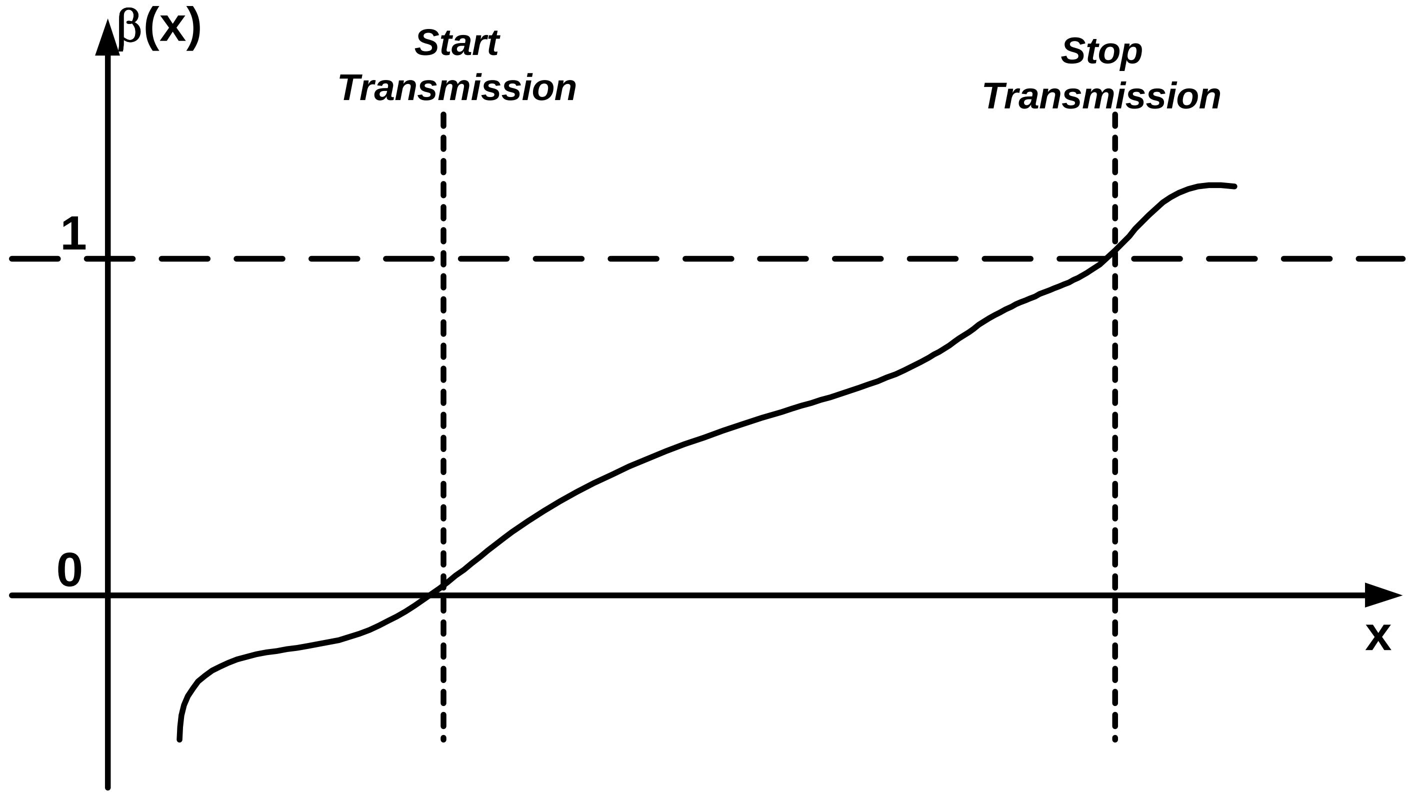
- Note that a "middle" receiver's rate depends only on its and its direct neighbors distance to the transmitter

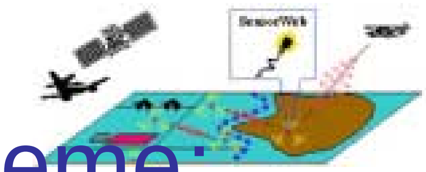
Optimal Communication Scheme: Large Networks



- Introduce a density of receivers
  - this needs to be done very carefully, since we are extending an inherently discrete concept
  - we use $p(x) = 1$ (i.e. one receiver in every Δx slice of the distance axis)
- Edge effects disappear
- Determine a "cumulative tradeoff function" $\beta(x)$
- Transmit to all receiver for x such that $\beta(x) \in [0,1]$
- The derivative of $\beta(x)$ - the "tradeoff function" $\alpha(x)$ - represents a power allocation strategy

Optimal Communication Scheme: Large Networks





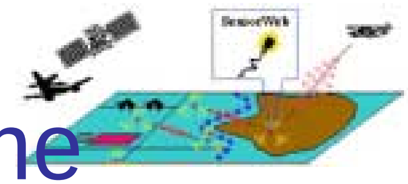
Optimal Communication Scheme: Large Networks

- $\beta(x)$ and $\alpha(x)$ do not depends on the density (as long as it has no holes)

$$\beta(x) = \frac{N}{P}\frac{\gamma - \rho}{\rho}x^{\gamma} \qquad \alpha(x) = \frac{N}{P}\gamma\frac{\gamma - \rho}{\rho}x^{\gamma - 1}$$

- Note that for $\gamma>1$, $\alpha(x)$ is an increasing function of x - thus the transport-capacity-optimizing power allocation scheme for such channels is to keep allocating more power to the receivers farther out until the power budget has been exhausted.

Optimal Communication Scheme Large Networks



- Transport capacity is then

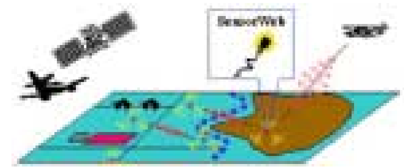
$$TC_{diffuse} = \frac{1}{2\ln 2}(\gamma - \rho)\int_0^{d^{uc}} x^{\rho-1} p(x)dx$$

- d^{uc} is the value of x at which $\beta(x) = 1$

$$d^{uc} = \sqrt[\gamma]{\frac{P}{N}\frac{\rho}{\gamma - \rho}}$$

- In the special case when $p(x)=1$ and $\rho=1$, we get

$$TC_{diffuse} = \frac{\gamma - 1}{2\ln 2}\sqrt[\gamma]{\frac{P}{N}\frac{1}{\gamma - 1}}$$



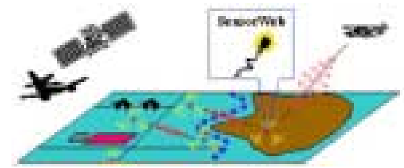
TDMA

- Problem: maximize

$$TC^{TDMA} = \sum_{k=1}^{K} \alpha_k d_k^{\rho} C\left(\frac{P}{N d_k^{\gamma}}\right)$$

- subject to

$$\sum_{k=1}^{K} \alpha_k = 1$$



TDMA

- Solution: transmit exclusively to the receiver whose distance maximizes

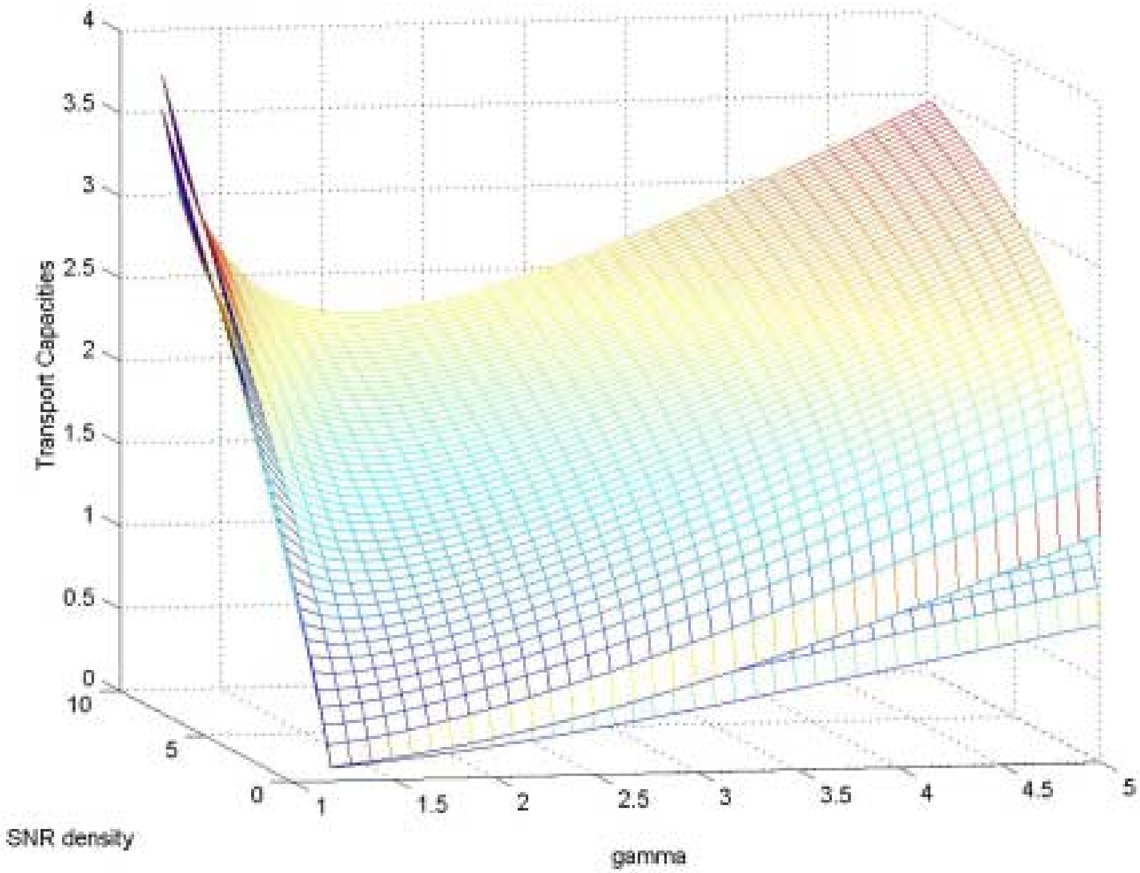
$$d_k^{\rho} C\left(\frac{P}{N d_k^{\gamma}}\right)$$

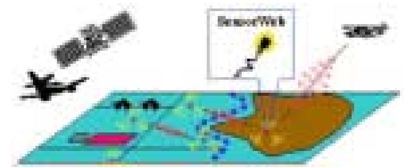
- Note: the "large network" solution is still to transmit to a single receiver with the optimal transport capacity given by

$$TC_{diffuse}^{TDMA} = \max_{d>0}\left(d^{\rho} C\left(\frac{P}{N d^{\gamma}}\right)\right)$$

- Note that the density p(*x*) does not affect the "large network" solution

Optimal vs. TDMA



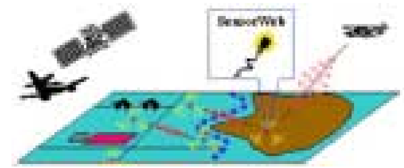


Power Law Model Problem

- Optimal transport capacity gets better as channel gets worse (as $\gamma \rightarrow \infty$)
- Due to a problem with the mode
  - as $d \rightarrow 0$, $SNR \rightarrow \infty$
  - the rate at which this happens increases as $\gamma \rightarrow \infty$
- Solution: change the model

$$SNR(d) = \frac{P}{Nd^{\gamma} + \tilde{N}}$$

- TDMA: little change - still transmit to a single receiver
- Optimal: need to redo the work



Modified Power Law Channel

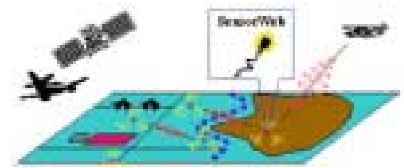
- Optimal Unconstrained β's are

$$\beta_k^* = \frac{N}{P}\frac{d_{k+1}^{\rho} d_k^{\rho}\left(d_{k+1}^{\gamma-\rho} - d_k^{\gamma-\rho}\right)}{d_{k+1}^{\rho} - d_k^{\rho}} - \frac{\tilde{N}}{P}$$

- Rates for "middle" receivers same as before

$$R_k = \frac{1}{2}\log_2\left(\frac{d_k^{\rho} - d_{k-1}^{\rho}}{d_{k+1}^{\rho} - d_k^{\rho}}\frac{d_{k+1}^{\gamma} - d_k^{\gamma}}{d_k^{\gamma} - d_{k-1}^{\gamma}}\right)$$

Modified Power Law Channel: Large networks

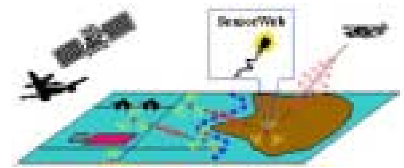


- Because the “middle” rates are the same - TC integral has the same integrand, different limits (β(*x*) is different)

$$TC_{diffuse} = \frac{1}{2\ln 2}(\gamma - \rho)\int_{d^{lc}}^{d^{uc}} x^{\rho-1} p(x)dx$$

- where

$$d^{uc} = \sqrt[\gamma]{\left(1+\frac{\tilde{N}}{P}\right)\frac{P}{N}\frac{\rho}{\gamma-\rho}} \qquad d^{lc} = \sqrt[\gamma]{\frac{\tilde{N}}{P}\frac{P}{N}\frac{\rho}{\gamma-\rho}}$$



Modified Power Law Channel: Large networks

- Now, for p(*x*) = 1 and ρ=1, we get

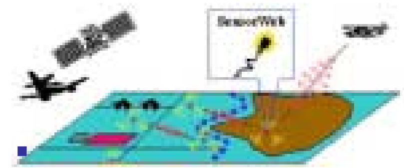
$$TC_{diffuse} = \frac{\gamma - 1}{2 \ln 2} \left(\sqrt[\gamma]{\left(1 + \frac{\tilde{N}}{P}\right)} - \sqrt[\gamma]{\frac{\tilde{N}}{P}} \right) \sqrt[\gamma]{\frac{P}{N} \frac{1}{\gamma - 1}}$$

- which, as $\gamma \to \infty$, converges to

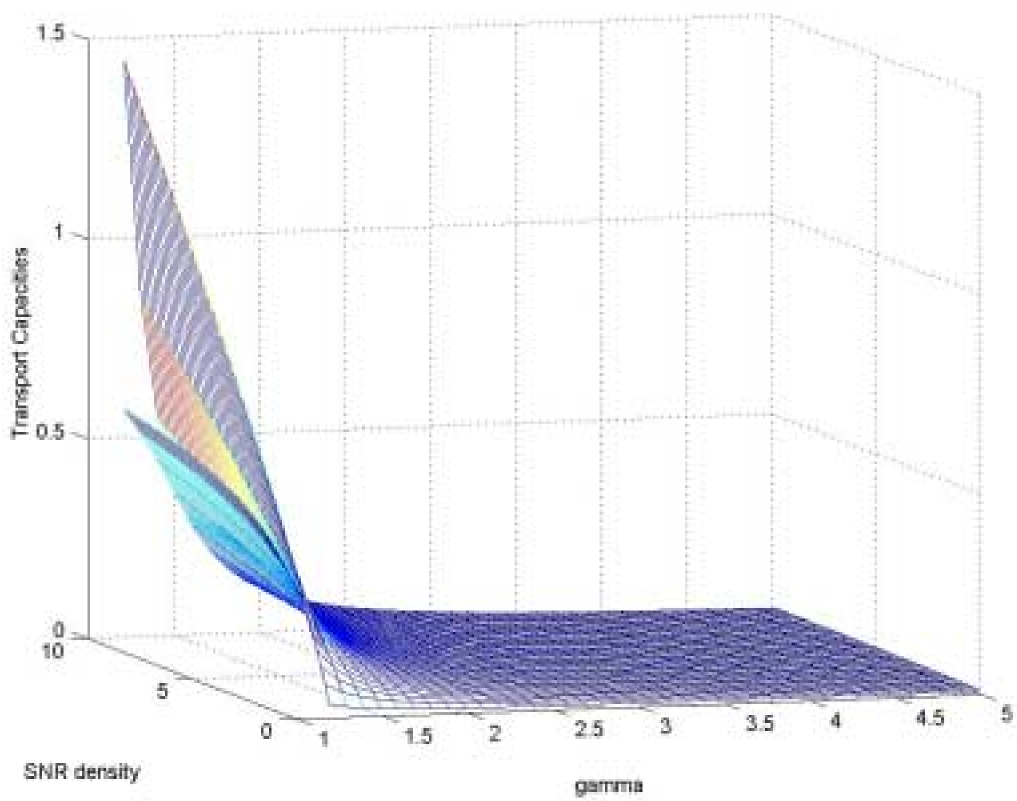
$$\lim_{\gamma \to \infty} TC_{diffuse} = \frac{1}{2} \log_2 \left(1 + \frac{P}{\tilde{N}} \right) = C\left(\frac{P}{\tilde{N}} \right)$$

- and

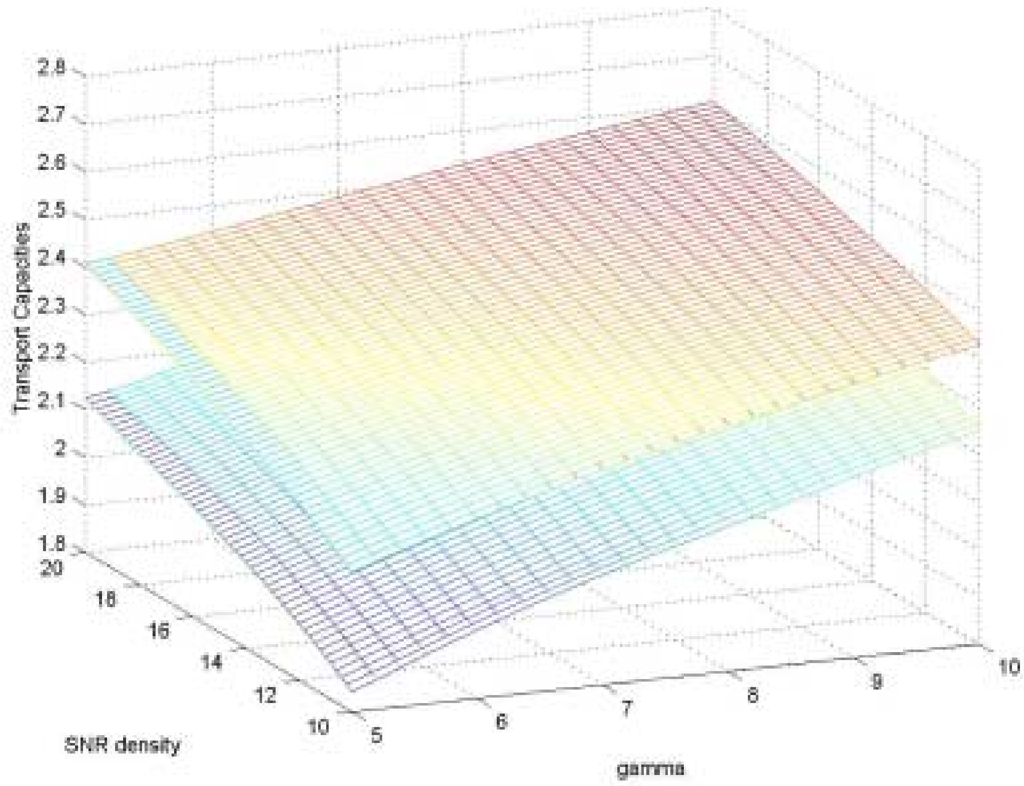
$$\lim_{\gamma \to \infty} TC_{diffuse} = \lim_{\gamma \to \infty} TC_{diffuse}^{TDMA}$$

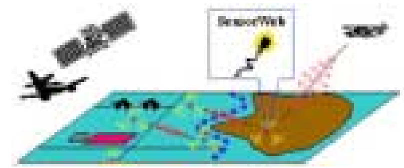


Modified Power Law Channel: Optimal vs. TDMA



Modified Power Law Channel: Optimal vs. TDMA

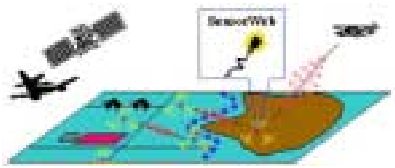




Summary and Conclusions

- For a broadcast Gaussian network, we established the transport-capacity maximizing communication strategy for a large set of reward and channel penalty functions

- We investigated the transport capacity of a large network for a power law channel and a modified power law channel for TDMA and optimal communication scheme



Directions for Future Work

- Similar work may be performed for other “pieces” of an ad-hoc network (e.g. multiple access channel) and other underlying channels. Feasibility greatly depends on how well the structure of the capacity region lends itself to analytical exploration
- More fundamentally, the Shannon capacity of a multiple-transmitter / multiple receiver network needs to be explored